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On the number of Courant-sharp Dirichlet eigenvalues

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In memory of Yuri Safarov

Abstract

We consider arbitrary open sets Ω in Euclidean space with finite Lebesgue measure, and obtain upper bounds for (i) the largest Courant-sharp Dirichlet eigenvalue of Ω, (ii) the number of Courant-sharp Dirichlet eigenvalues of Ω. This extends recent results of P. Bérard and B. Helffer.

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1 Introduction

Let Ω be an open set in Euclidean space $\mathbb{R}^m$ with finite Lebesgue measure $|\Omega|$ and boundary $\partial\Omega$. We denote the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ by $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \lambda_3(\Omega) \le \dots$ taking the multiplicities of these eigenvalues into account. We define the counting function for Ω by

$$N_\Omega(\lambda) = \sharp\{n \in \mathbb{N} : \lambda_n(\Omega) < \lambda\}.$$

Weyl's law asserts that

$$N_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} + o(\lambda^{m/2}),\ \lambda \to \infty, \tag{1}$$

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where ω_m is the measure of a ball $\mathcal{B}_m$ with radius 1 in $\mathbb{R}^m$. We refer to Theorem 2 in [16] for a proof of (1) in this generality. For a proof of Weyl's law with a non-trivial remainder estimate for Ω open, bounded and connected we refer to Theorem 1.8 in [12].

Let $\{\varphi_{1,\Omega}, \varphi_{2,\Omega}, \dots\}$ be an orthonormal basis in the Sobolev space $H_0^1(\Omega)$ of eigenfunctions corresponding to the Dirichlet eigenvalues. These eigenfunctions satisfy the Dirichlet boundary conditions in the usual trace sense. Let $\nu(\varphi_{n,\Omega})$ denote the number of nodal domains of $\varphi_{n,\Omega}$. Then Pleijel's theorem ([13]) states that

$$\limsup_{n\to\infty} \frac{\nu(\varphi_{n,\Omega})}{n} \le \gamma_m,$$

where

$$\gamma_m = \frac{(2\pi)^m}{\omega_m^2} \big(\lambda_1(\mathcal{B}_m)\big)^{-m/2} < 1. \tag{2}$$

It is known that Pleijel's bound is not sharp. See [7], [18] and [14].

We say that $\lambda_n(\Omega)$ is Courant-sharp if $\nu(\varphi_{n,\Omega}) = n$. Courant's nodal domain theorem asserts that $\nu(\varphi_{n,\Omega}) \le n$. Courant's original proof in [8] was for the planar case. This has been subsequently stated and proved in a Riemannian manifold setting in [3]. See also [13]. Pleijel's theorem implies that for a given Ω the number of Courant-sharp Dirichlet eigenvalues is finite. Using results of [5] and [17], Bérard and Helffer, [1], obtained an upper bound for the largest Courant-sharp Dirichlet eigenvalue if Ω is bounded and has smooth boundary $\partial\Omega$.

This paper concerns arbitrary open sets in $\mathbb{R}^m$ with finite Lebesgue measure. The proofs of Courant's theorem in [8], [13] and [3] all use the fact that a restriction of an eigenfunction to a nodal domain U is the first Dirichlet eigenfunction on U. This is immediate if $(\partial\Omega) \cap (\partial U)$ is sufficiently regular. The above fact holds without that regularity requirement. See for example Theorem 1.1 in [9].

Our main result, Theorem 1 below is for open sets Ω in $\mathbb{R}^m$ with finite Lebesgue measure. We obtain (i) an upper bound for the largest Dirichlet eigenvalue of Ω which is Courant-sharp, and (ii) an upper bound for the number of Courant-sharp eigenvalues of Ω. For $A \subset \mathbb{R}^m, A \ne \emptyset$ let

$$d(x, A) = \inf\{|x - y| : y \in A\}.$$

For $\epsilon \ge 0$ and $|\Omega| < \infty$ we define

$$\mu_\Omega(\epsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}|,$$

and

$$\epsilon(\Omega) = \inf\{\epsilon : \mu_\Omega(\epsilon) \ge 2^{-1}(1 - \gamma_m)|\Omega|\}. \tag{3}$$

We denote the number of Courant-sharp eigenvalues of Ω by $\mathfrak{C}(\Omega)$.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^m$ with finite Lebesgue measure. We have the following.*

(i) *If $\lambda_n(\Omega)$ is Courant-sharp then*

$$\lambda_n(\Omega) \le \left(\frac{2\pi m^2}{(1 - \gamma_m)\epsilon(\Omega)}\right)^2. \tag{4}$$

(ii)

$$\mathfrak{C}(\Omega) \le \frac{\omega_m}{(1-\gamma_m)^m}\big(m^3(m+2)\big)^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}. \tag{5}$$

(iii) *If $n \in \mathbb{N}, n > \frac{\omega_m}{(1-\gamma_m)^m}\big(m^3(m+2)\big)^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}$, then $\lambda_n(\Omega)$ is not Courant-sharp.*

In Section 2 below we prove Theorem 1. In Section 3 we analyse some examples including the von Koch snowflake.

2 Proof of Theorem 1

Suppose $\lambda_n(\Omega)$ is Courant-sharp with eigenfunction $\varphi_{n,\Omega}$. Let $U_1, \dots, U_n$ be the nodal domains of $\varphi_{n,\Omega}$ so that $\lambda_n(\Omega) = \lambda_1(U_1) = \cdots = \lambda_1(U_n)$. Without loss of generality we may assume that $|U_1| \le |U_2| \le \cdots \le |U_n|$. Hence $|U_1| \le |\Omega|/n$. By Faber-Krahn we have that

$$\lambda_n(\Omega) = \lambda_1(U_1) \ge \lambda_1(\mathcal{B}_m)\left(\frac{n\omega_m}{|\Omega|}\right)^{2/m}.$$

It follows that, since $\lambda_{n-1}(\Omega) < \lambda_n(\Omega)$,

$$\begin{aligned}(\lambda_n(\Omega))^{m/2} &\ge (\lambda_1(\mathcal{B}_m))^{m/2}\frac{n\omega_m}{|\Omega|}\\ &\ge (\lambda_1(\mathcal{B}_m))^{m/2}\frac{\omega_m}{|\Omega|}(n-1)\\ &= (\lambda_1(\mathcal{B}_m))^{m/2}\frac{\omega_m}{|\Omega|}N_\Omega(\lambda_n(\Omega)).\end{aligned}$$

This gives that

$$\frac{\omega_m}{(2\pi)^m}(1-\gamma_m)|\Omega|(\lambda_n(\Omega))^{m/2} \le R_\Omega(\lambda_n(\Omega)), \tag{6}$$

where $R_\Omega : \mathbb{R}^+ \mapsto \mathbb{R}$ is defined by

$$R_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} - N_\Omega(\lambda). \tag{7}$$

See (15) and (16) in [1]. Below we obtain an upper bound for $R_\Omega(\lambda)$. Let $\epsilon > 0$ be arbitrary. Consider the collection $\mathfrak{M}_\epsilon$ of open cubes of measure ϵ^m with vertices in the set of m-tuples $\{\mathbb{Z}\epsilon, \dots, \mathbb{Z}\epsilon\}$. Let $M_\Omega(\epsilon)$ be the number of open cubes of side-length ϵ in $\mathfrak{M}_\epsilon$ which are contained in Ω,

$$M_\Omega(\epsilon) = \sharp\{N \in \mathfrak{M}_\epsilon : N \subset \Omega\}.$$

We have that

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \ge 0. \tag{8}$$

In order to obtain an upper bound for the left hand-side of (8) we let $x \in \Omega$. If $d(x, \partial\Omega) > m^{1/2}\epsilon$, then x belongs to an open ϵ-cube in $\mathfrak{M}_\epsilon$ contained in Ω.

Hence the measure of the set which is not covered by the ϵ-cubes in $\mathfrak{M}_\epsilon$ that are entirely contained in Ω is bounded from above by $\mu_\Omega(m^{1/2}\epsilon)$. So

$$|\Omega| - M_\Omega(\epsilon)\epsilon^m \le \mu_\Omega(m^{1/2}\epsilon). \tag{9}$$

By Dirichlet bracketing (see [15]) we have that

$$N_\Omega(\lambda) \ge M_\Omega(\epsilon) N_{C_\epsilon}(\lambda), \tag{10}$$

where C_ϵ is an open cube in $\mathbb{R}^m$ with side-length ϵ. The following standard estimate is attributed to Gauss:

$$\begin{aligned} N_{C_\epsilon}(\lambda) &= \sharp\left\{(k_1,\dots,k_m)\in\mathbb{N}^m : \sum_{i=1}^m k_i^2 < \pi^{-2}\epsilon^2\lambda\right\} \\ &\ge \frac{\omega_m}{2^m}\left(\pi^{-1}\epsilon\lambda^{1/2} - m^{1/2}\right)_+^m \\ &\ge \frac{\omega_m}{(2\pi)^m}\epsilon^m\lambda^{m/2}\left(1 - \frac{\pi m^{3/2}}{\epsilon\lambda^{1/2}}\right), \end{aligned} \tag{11}$$

where $+$ denotes the positive part. By (10) and (11),

$$\begin{aligned} N_\Omega(\lambda) &\ge M_\Omega(\epsilon)N_{C_\epsilon}(\lambda) \\ &\ge M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\epsilon^m\lambda^{m/2} - M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\pi m^{3/2}\epsilon^{m-1}\lambda^{(m-1)/2} \\ &= \frac{\omega_m}{(2\pi)^m}|\Omega|\lambda^{m/2} - (|\Omega| - M_\Omega(\epsilon)\epsilon^m)\frac{\omega_m}{(2\pi)^m}\lambda^{m/2} \\ &\qquad - M_\Omega(\epsilon)\frac{\omega_m}{(2\pi)^m}\pi m^{3/2}\epsilon^{m-1}\lambda^{(m-1)/2}. \end{aligned} \tag{12}$$

We bound the second and third terms in the right hand-side of (12) using (9) and (8) respectively. This then gives, by (7), that

$$R_\Omega(\lambda) \le \frac{\omega_m}{(2\pi)^m}\mu_\Omega(m^{1/2}\epsilon)\lambda^{m/2} + \frac{\pi m^{3/2}\omega_m}{(2\pi)^m}\frac{|\Omega|\lambda^{(m-1)/2}}{\epsilon}. \tag{13}$$

By (6) and (13) we have that if $\lambda_n(\Omega)$ is Courant-sharp then

$$\begin{aligned} \frac{\omega_m}{(2\pi)^m}(1-\gamma_m)|\Omega|(\lambda_n(\Omega))^{m/2} &\le \frac{\omega_m}{(2\pi)^m}\mu_\Omega(m^{1/2}\epsilon)(\lambda_n(\Omega))^{m/2} \\ &\qquad + \frac{\pi m^{3/2}\omega_m}{(2\pi)^m}\frac{|\Omega|(\lambda_n(\Omega))^{(m-1)/2}}{\epsilon}. \end{aligned} \tag{14}$$

We now choose ϵ such that the second term in the right hand-side of (14) equals half of the left hand-side of (14). That is

$$\epsilon = 2\pi m^{3/2}(1-\gamma_m)^{-1}(\lambda_n(\Omega))^{-1/2}. \tag{15}$$

By (14) and the choice of ϵ in (15) we have that if $\lambda_n(\Omega)$ is Courant-sharp then

$$2^{-1}(1-\gamma_m)|\Omega| \le \mu_\Omega(2\pi m^2(1-\gamma_m)^{-1}(\lambda_n(\Omega))^{-1/2}). \tag{16}$$

Since $\epsilon \mapsto \mu_\Omega(\epsilon)$ is continuous and onto $[0,|\Omega|]$ the infimum in (3) is over a non-empty set which is bounded from below, and therefore exists. So if $\lambda_n(\Omega)$

is Courant-sharp then, by (3) and (16), $\frac{2\pi m^2}{(1-\gamma_m)(\lambda_n(\Omega))^{1/2}} \ge \epsilon(\Omega)$. This proves Theorem 1(i).

By [11] we also have that

$$\lambda_n(\Omega) \ge \frac{m}{m+2}\frac{(2\pi)^2}{\omega_m^{2/m}}\left(\frac{n}{|\Omega|}\right)^{2/m}. \tag{17}$$

This, together with (4), implies (5) and proves Theorem 1(ii).

To prove Theorem 1(iii) we just note that by (17),

$$\max\left\{n \in \mathbb{N} : \lambda_n(\Omega) \le \left(\frac{2\pi m^2}{(1-\gamma_m)\epsilon(\Omega)}\right)^2\right\} \le \frac{\omega_m}{(1-\gamma_m)^m}\big(m^3(m+2)\big)^{m/2}\frac{|\Omega|}{\epsilon(\Omega)^m}.$$

□

We note that if we were to use the lower bounds for the counting function from Section 2 in [5] then we would have to assume a weak integrability condition on μ_Ω of the form $\int \epsilon^{-1}\, d\mu_\Omega(\epsilon) < \infty$. Such an integrability condition may fail if the interior Minkowski dimension of $\partial\Omega$ is equal to m. The procedure above avoids this integrability condition.

3 Examples

In this section we analyse three examples where explicit computations seem out of reach.

Example 1. *Let Ω be an open, bounded, convex set in $\mathbb{R}^m$. Let $\mathcal{H}^{m-1}(\partial\Omega)$ denote the $(m-1)$-dimensional Hausdorff measure of $\partial\Omega$. Then*

$$\mathfrak{C}(\Omega) \le \frac{\omega_m}{(1-\gamma_m)^{2m}}\big(4m^3(m+2)\big)^{m/2}\frac{\big(\mathcal{H}^{m-1}(\partial\Omega)\big)^m}{|\Omega|^{m-1}}. \tag{18}$$

Proof. By convexity of Ω we have that

$$\mu_\Omega(\epsilon) \le \mathcal{H}^{m-1}(\partial\Omega)\epsilon.$$

By (3),

$$\epsilon(\Omega) \ge 2^{-1}(1-\gamma_m)\frac{|\Omega|}{\mathcal{H}^{m-1}(\partial\Omega)}, \tag{19}$$

and (18) follows from Theorem 1 and (19). □

It was shown in [10] that only the first, second and fourth Dirichlet eigenvalues for $\mathcal{B}_2$ are Courant-sharp. Hence $\mathfrak{C}(\mathcal{B}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{B}_2$ is equal to $j_{0,2}^2$. Here $j_{0,2} \asymp 5.520..$ is the second positive zero of the Bessel function J_0. A straightforward computation using (4) and (19) shows that the largest Courant-sharp eigenvalue of $\mathcal{B}_2$ is strictly less than $1.2\cdot 10^6$. This compares well with the bound $7.1\cdot 10^6$ obtained in [1]. For the unit square $\mathcal{C}_2$ it is known ([13], [2]) that only the first, second and fourth Dirichlet eigenvalues are Courant-sharp. Hence $\mathfrak{C}(\mathcal{C}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{C}_2$ is equal to $8\pi^2$. Using (4) and (19) we have that the largest Courant-sharp eigenvalue of the unit square is strictly less than $4.5\cdot 10^6$, whereas

[1] gives a bound $5.9 \cdot 10^6$. These examples illustrate that the bounds obtained in Theorem 1 are very crude.

The second example is a von Koch snowflake K with similarity ratio $\frac{1}{3}$. We recall its construction. Let the basic square (generation 0) in K have side-length 1. The first generation consists of 4 squares with side-length $\frac{1}{3}$ each attached symmetrically to the basic square. Proceeding inductively we have that the j'th generation in K, $j \in \mathbb{N}$ consists of $4 \cdot 5^{j-1}$ squares with side-length 3^{-j}. We let K be the interior of its closure. Then K is connected, has Lebesgue measure $|K| = 2$, and both the Hausdorff dimension of ∂K and the interior Minkowski dimension of ∂K are equal to $\log 5/\log 3$. See Figure 1, and [4] for further details.

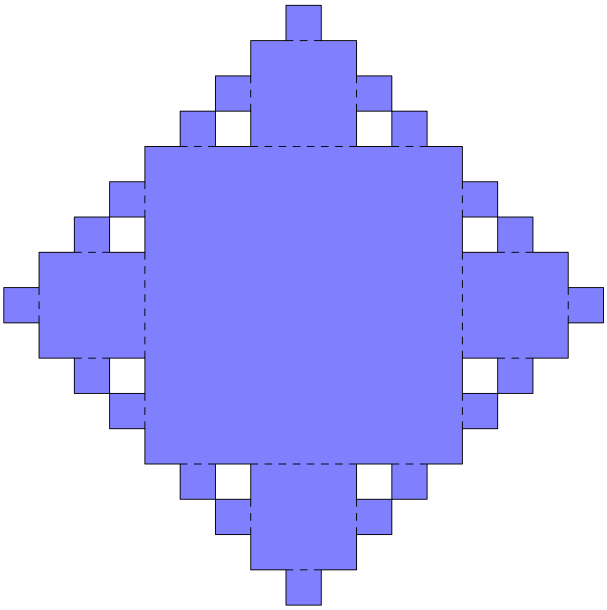

Figure 1: The first two generations of K

Example 2. *Let K be the von Koch snowflake generated by the unit square and similarity ratio $\frac{1}{3}$. Then*

$$\mathfrak{C}(K) \leq 15 \cdot 10^7. \tag{20}$$

Proof. By Theorem 1, (2), and $|K| = 2$ we find that

$$\mathfrak{C}(K) \leq \frac{64\pi j_0^4}{(j_0^2 - 4)^2} \epsilon(K)^{-2}, \tag{21}$$

where we have used that

$$\lambda_1(\mathcal{B}_2) = j_0^2,$$

where $j_0 = 2.405...$ is the first positive zero of the Bessel function J_0. It remains to find a lower bound for $\epsilon(K)$. We obtain an upper bound for $\mu_\Omega(\epsilon)$ by adding all edges between squares of different generations. This gives a disjoint union of 1 unit square and $4 \cdot 5^{j-1}$ squares with side-lengths $3^{-j}, j \in \mathbb{N}$. Let $\epsilon < \frac{1}{18}$, and let $J \in \mathbb{N}$ be such that

$$J < \frac{\log\left(\frac{1}{2\epsilon}\right)}{\log 3} \le J + 1.$$

Then $J \ge 2$. The contribution to the upper bound for $\mu_\Omega(\epsilon)$ from the squares in generations $1, \dots, J-1$ is bounded from above by

$$\left(4 + 16 \sum_{j=1}^{J-1} 5^{j-1} 3^{-j}\right)\epsilon \le \frac{24\epsilon}{5}\left(\frac{5}{3}\right)^J \le \frac{48}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}}. \tag{22}$$

The first term in the left-hand side above is the contribution from the unit square. The contribution to the upper bound for $\mu_\Omega(\epsilon)$ from the squares in generations $J, J+1, \dots$ is bounded from above by

$$\sum_{j \ge J} 4 \cdot 5^{j-1} 9^{-j} = \left(\frac{5}{9}\right)^{J-1} \le \frac{36}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}}. \tag{23}$$

We recognise the interior Minkowski dimension $\frac{\log 5}{\log 3}$ of ∂K. By (22) and (23) we have that

$$\mu_\Omega(\epsilon) \le \frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}},\ 0 < \epsilon < \frac{1}{18}.$$

Solving the equation

$$\frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon^{2-\frac{\log 5}{\log 3}} = 1 - \frac{4}{j_0^2}$$

gives that

$$\epsilon(K) \ge 0.00379. \tag{24}$$

The bound of (20) follows by (21) and (24). □

Below we construct an open set $D_s \subset \mathbb{R}^3$. Let $Q_0 \subset \mathbb{R}^3$ be an open cube of side-length 1. Let $0 < s \le \sqrt{2} - 1$. Attach a regular open cube $Q_{1,i}$ of side-length s to the centre $c_{1,i}, i = 1, \dots, 6$, of each face of ∂Q_0, and such that all the faces are pairwise-parallel. Now proceed by induction. For $j = 2, 3, \dots,$ attach $N(j) = 6 \cdot 5^{j-1}$ open cubes $Q_{j,1}, \dots, Q_{j,N(j)}$, of side-length s^j to the centres of the boundary faces of the cubes $Q_{j-1,1}, \dots, Q_{j-1,N(j-1)}$, again with pairwise-parallel faces. We define the polyhedron D_s as

$$D_s = \text{interior}\left\{\overline{Q_0 \cup [\bigcup_{j \ge 1} \bigcup_{1 \le i \le N(j)} Q_{j,i}]}\right\}.$$

See Figure 2. We note that for $0 < s \le \sqrt{2} - 1$ no cubes in the construction of D_s overlap.

The asymptotic behaviour of the heat content of D_s in $\mathbb{R}^3$ for small time was analysed in [6]. Here we have the following.

Example 3. *Let $s \in (0, \sqrt{2} - 1]$, and let D_s be the polyhedron in $\mathbb{R}^3$ defined above. Then*

$$\mathfrak{C}(D_s) \le 25 \cdot 10^{10}. \tag{25}$$

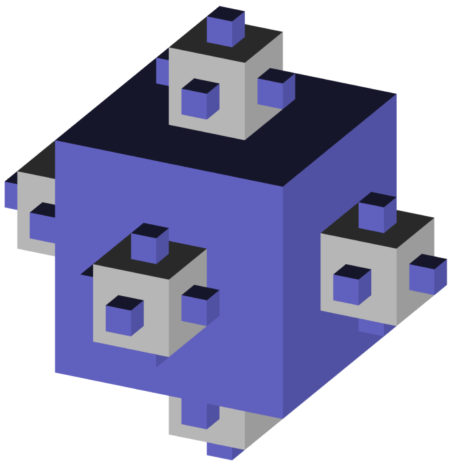

Figure 2: The first two generations of D_s with $s = \frac{1}{3}$.

Proof. We have that

$$|D_s| = \frac{1+s^3}{1-5s^3},$$

and that the two-dimensional Hausdorff measure of the boundary is given by

$$\mathcal{H}^2(\partial D_s) = 6\left(\frac{1-s^2}{1-5s^2}\right).$$

By Theorem 1 we have that

$$\mathfrak{C}(D_s) \le 36(15)^{3/2}\pi\left(1-\frac{9}{2\pi^2}\right)^{-3}\frac{|D_s|}{\epsilon(D_s)^3}, \tag{26}$$

where we have used that

$$\lambda_1(\mathcal{B}_3) = j_{1/2}^2 = \pi^2,$$

where $j_{1/2} = \pi$ is the first positive zero of the Bessel function $J_{1/2}$. We obtain an upper bound for $\mu_\Omega(\epsilon)$ by adding all faces between cubes of different generations. This gives a disjoint union of 1 unit cube and $6 \cdot 5^{j-1}$ cubes of side-length s^j, $j \in \mathbb{N}$. Hence

$$\mu_\Omega(\epsilon) \le \left(6 + 36\sum_{j=1}^{\infty} 5^{j-1}s^{2j}\right)\epsilon = \frac{6(1+s^2)}{1-5s^2}\epsilon. \tag{27}$$

By (3) and (27) we have that

$$\epsilon(D_s) \ge \frac{1}{12}\left(1-\frac{9}{2\pi^2}\right)\frac{1-5s^2}{1+s^2}|D_s|. \tag{28}$$

Finally by (26), (28), the fact that $0 < s \le \sqrt{2}-1$, and $|D_s| \ge 1$ we obtain that

$$\mathfrak{C}(D_s) \le 6(12)^4(15)^{3/2}(140+99\sqrt{2})\pi\left(1-\frac{9}{2\pi^2}\right)^{-6}.$$

This implies (25). $\square$

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